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On discontinuous strain fields in finite elastostatics

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Abstract

A general method for the study of piece-wise homogeneous strain fields in finite elasticity is proposed. Critical homogeneous deformations, supporting strain jumping, are defined for any anisotropic elastic material under constant Piola–Kirchhoff stress field in three-dimensional elasticity. Since Maxwell's sets appear in the neighborhood of singularities higher than the fold, the existence of a cusp singularity is a sufficient condition for the emergence of piece-wise constant strain fields. General formulae are derived for the study of any problem without restrictions or fictitious stress–strain laws. The theory is implemented in a simple shearing plane strain problem. Nevertheless, the procedure is valid for any anisotropic material and three-dimensional problems.

Keywords: Continuum mechanics; Large deformations; Stability; Bifurcation; Singularities; Two-phase strain; Maxwell's sets

1. Introduction

Two-phase deformations in solid materials have recently received much attention. Various models have been provided (Ericksen, 1991; Truskinovsky and Zanzotto, 1996; Knowles and Sternberg, 1978; Abeyaratne, 1980) for studying the behavior of twinning in crystals (Pitteri and Zanzotto, 2003) austenite–martensite transformations in certain alloys that occur in shape-memory alloys (Khatchaturyan, 1983) and ferroelastic materials (Salze, 1990).

Knowles and Sternberg (1978) and Knowles and Sternberg (1977) attribute the emergence of discontinuous strain fields to the loss of ellipticity. Furthermore, some conditions have been derived for the development of piece-wise homogeneous deformations in compressible materials (Rosakis, 1990; Rosakis and

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Jiang, 1993). Ericksen (1991), adopting globally stable criteria. Fosdick and MacSithigh (1986) have defined conditions for the emergence of two-phase deformations in incompressible materials. Further, Fosdick and MacSithigh (1991) study the problem in inextensible materials.

However, following the existing theories, the study of the emergence of stepwise homogeneous strain, under uniform stress in any anisotropic material, with possible multiple constraints, is a quite hard problem. In many cases the various problems are reduced to one-dimensional ones with fictitious constitutive laws. The present work has the ambition to introduce a new technique for the solution of a two-phase problem of a initially homogeneous deformation using singularity theory. The procedure may be extended to any anisotropic material with possible existence of incompressibility or inextensibility constraints. The idea is quite simple.

First the stability of homogeneous deformations is studied and the critical states (Vainberg and Tenogin, 1974; Thompson and Hunt, 1973) are located satisfying a constraint required by a possible jumping of the displacement gradient (James, 1981; Gurtin, 1983). It turns out that this constraint depends only on the kernel of the critical state. Classification of the various singularities (Gibson, 1979; Thom, 1975; Zeeman, 1977; Gilmore, 1981) of the total potential energy for homogeneous deformations has already been performed (Lazopoulos and Markatis, 1996). It is well known that globally stable states are developed on Maxwell's sets in the neighborhood of the various singularities (Gilmore, 1981). Furthermore Maxwell's sets are developed in the neighborhood of singularities higher than the fold. Therefore the existence of a cusp is a sufficient condition for the emergence of discontinuous strain fields. Using the unfolding in the neighborhood of the cusp, the discontinuous field is completely defined along with the phase boundary. The procedure is quite general. Simple formulae will be found. Direct results handily may be derived, using computerized algebra packs, such as Mathematica (Wolfram, 1996) or Maple. The present method is restricted only to homogeneous deformations anisotropic compressible materials under any kind of suitable stress traction. No restrictions are imposed upon the material and the constitutive laws either. Conventional similar problems in incompressible materials with fictitious constitutive laws have been studied by De Tommasi et al. (2001) and DD' Ambrosio et al. (2003).

First the one-dimensional (bar) problem is studied, where the two phase strain field under uni-axial strain is described with the help of Maxwell's set in the neighborhood of the cusp singularity. Next the three-dimensional problem is analyzed. The cusp singularity is located, in the class of incremental deformations satisfying the jumping condition. The corresponding Maxwell's set helps in defining two global minima describing the piece-wise homogeneous strain field under homogeneous stress.

The theory is implemented in a simple shearing problem of Blatz and Ko (1962) material explaining the various steps. Although the material for the application is isotropic, the method works for any anisotropic homogeneous material and three-dimensional problems as well. Similar materials have been invoked by Knowles and Sternberg (1978, 1977); Mathematica computerized algebra pack, has been applied for deriving the various formulae and performing the computing as well.

2. Discontinuous strain fields in the one-dimensional case

Ericksen (1991, 1975) introduced the coexistence of phases phenomena in solids, developing globally stable equilibrium configurations. Adapting, likewise, globally (Maxwell) stability criteria, two-phase equilibrium configurations were emerged allowing for continuous displacement but discontinuous strain fields. Ericksen (1991, 1975), in fact, adopted, non-convex strain energy density function $W(u'(x))$ for the uni-axial tension equilibrium deformation of a bar, where $u(x)$ is the axial displacement and $(') = \frac{\mathrm{d}()}{\mathrm{d}x}$ as usual, see Fig. 1. Therefore, the potential energy function per unit length of the bar is defined by,

$$V = W(u'(x)) - \sigma u'(x) \tag{1}$$

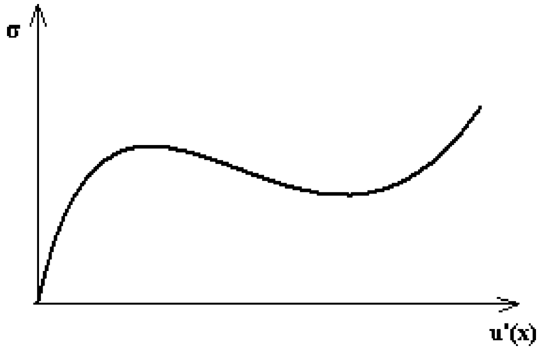

Fig. 1. The non-convex stress–strain diagram.

where, σ is the first Piola–Kirchhoff axial stress. The variational problem for the functional V in the C^0 class of displacements $u(x)$, i.e. continuous displacement with discontinuous strains, is given by,

$$\sigma = \frac{\partial W}{\partial u'} \tag{2}$$

and the corner, Erdmann–Weierstrass condition (Gelfand and Fomin, 1963) at the point x_0 of discontinuity,

$$\left[\frac{\partial W}{\partial u'}\right]\Bigg|_{x_0} = 0 \tag{3}$$

$$[W - \sigma u']|_{x_0} = 0 \tag{4}$$

where, $[.] = (.)|_{x_{0+}} - (.)|_{x_{0-}}$ denotes the jumping at the point of discontinuity as usual. Equilibrium Eq. (2) and the corner conditions, Eqs. (3) and (4), reveal non-unique globally stable equilibrium deformations if the Piola–Kirchhoff stress σ reaches a value σ_{M}, that is called Maxwell's, value intersecting the stress–strain curve and cutting off equal areas A and B, see Fig. 2. Therefore the bar is allowed to develop piece-wise constant strain fields, exhibiting the coexistence of phases phenomenon; Sometimes the distribution is quite fine emerging the twinning of crystal phenomena (Ball and James, 1987; Pitteri and Zanzotto, 2003).

Nevertheless just the same problem may be studied in the context of singularity theory. Since the simplest (lowest) singularity including Maxwell's sets is the cusp catastrophe (Gilmore, 1981) the present discussion is limited to that singularity. Higher singularities include anyway Maxwell's sets and may be useful

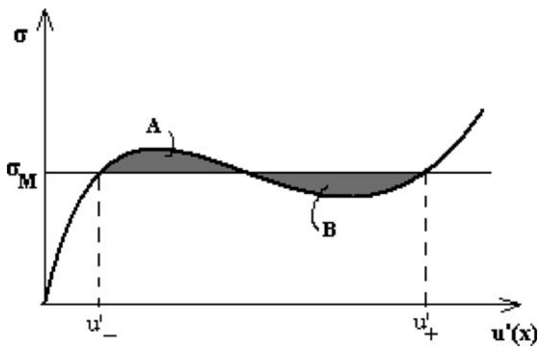

Fig. 2. Maxwell's stress and strains.

for many other problems with many (more than two) global minima. Indeed, the total potential energy V is expressed in this case by,

$$V = s^4 + as^2 + bs \tag{5}$$

Just exploring the control space (a, b) of the cusp singularity, see Fig. 3, the stable regions A_1 with unique minimum of the total energy function are prescribed by the fold curve A_2 where multiple local extremals are shown up. Further, the fold curve is described by the relation,

$$8a^3 + 27b^2 = 0 \tag{6}$$

This is, also, the critical curve for locally stable transitions. Nevertheless, the set for globally stable transitions, called Maxwell's set, is the semi-axis in the control space with $a < 0$, see Fig. 3. In that case equilibrium states with,

$$\xi = \pm\sqrt{-\frac{2a}{3}} \tag{7}$$

yield global minima of the total energy function. Consequently, if the strain of the bar is equal to the constant u'_0 when the extension stress is equal to σ_0 and the stress is increased by $\mathrm{d}\sigma$, then the strain changes by $\mathrm{d}u'(x)$. Let us assume that the strain energy density function in the neighborhood of the equilibrium state (u'_0, σ_0) is defined by a cusp unfolding,

$$W(u'(x)) = \mathrm{d}u'(x)^4 - a_0\mathrm{d}u'(x)^2 + b_0\mathrm{d}u'(x) + W(u'_0) \tag{8}$$

with $a_0, b_0 > 0$. Consequently Maxwell's sets will be met in the neighborhood of the cusp if

$$\sigma_0 = b_0 \tag{9}$$

In this case

$$\mathrm{d}u'(x) = \pm\sqrt{\frac{2a_0}{3}}$$

and the two phases are defined by,

$$u'(x) = u'_0 \pm \sqrt{\frac{2a_0}{3}} \tag{10}$$

Therefore, there are regions of piece-wise constant distributed strain in the bar, sometimes so fine that macroscopically are viewed as a third strain (Ball and James, 1987).

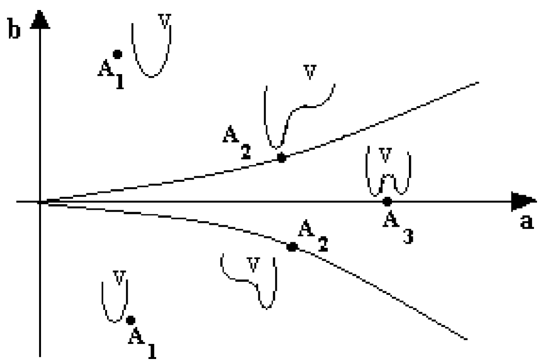

Fig. 3. Geometry of the cusp control space.

3. The cusp condition in compressible homogeneous deformations

The method, presented in the preceding section, will be extended into a three-dimensional case. Indeed, piece-wise homogeneous deformations, under the action of surface traction, will be studied. Transferring the simple preceding method into the three-dimensional case is not a simple matter. Two major problems are shown up in that case. First, the location of the cusp singularities of the total potential energy function with many independent variables and secondly, the compatibility of the deformation gradients for the emergence of the jumping for the strain. The interaction of those two factors will be studied below.

Lazopoulos and Markatis (1996), have recently presented the classification of the singularities of the potential energy function for homogeneous deformations of any elastic material with (or without) the interaction of any (multiple) constraints. Classification of the simple singularities (cuspoids) will be performed in the class of deformations, satisfying the deformation gradient jump compatibility conditions. Consider an anisotropic hyperelastic material under homogeneous deformation defined by the displacement vector,

$$u = (u_1, u_2, u_3) \tag{11}$$

The non-linear strains are expressed by (Green and Atkins, 1970),

$$e_{ij} = \frac{1}{2}(u_{ij} + u_{ji} + u_{ri}u_{rj}) \tag{12}$$

with $u_{ij} = \frac{\partial u_i}{\partial x_j}$. Likewise, the strain energy density W is defined as a function of the strains,

$$W = W(e_{ij}) \tag{13}$$

Since the deformation is homogeneous, the potential energy density function V is expressed by,

$$V = W(e_{ij}) - t_{ij}u_{ij} \tag{14}$$

where, t_{ij} are the components of the first Piola–Kirchhoff stress tensor T, referred to the unstressed reference placement (Gurtin, 1981). In addition the deformation gradient is defined by,

$$F_{ij} = \delta_{ij} + u_{ij} \tag{15}$$

It is evident that the total potential density function V is a function of nine components of the displacement gradient u_{ij}. Nevertheless, the conservation of rotational momentum requiring,

$$TF^{\mathrm{T}} = FT^{\mathrm{T}} \tag{16}$$

restricts the number of variables from nine to six. In case the system (16) is solvable, it may be solved for three variables, let us say, u_{21}, ,u_{32}, u_{31} and substituting into the potential energy density function V we get,

$$V = V(q_i, t_k) = V(u_{ab}, t_{cd}), \quad i,k = 1,\ldots,6, \ \ a,b,c,d = 1,2,3 \tag{17}$$

where,

$$q_1 = u_{11}, \quad q_2 = u_{22}, \quad q_3 = u_{33}, \quad q_4 = u_{12}, \quad q_5 = u_{13}, \quad q_6 = u_{23} \tag{18a}$$

and t_i, $i = 1, \ldots, 9$ with,

$$t_1 = t_{11}, \quad t_2 = t_{22,} \quad t_3 = t_{33}, \quad t_4 = t_{12,} \quad t_5 = t_{13}, \quad t_6 = t_{23,} \quad t_7 = t_{21}, \quad t_8 = t_{31,} \quad t_9 = t_{32} \tag{18b}$$

Hence,

$$\nabla V = V_i(q_j, t_k) = \frac{\partial V}{\partial q_i} = 0, \quad i,j = 1,\ldots,6, \quad k = 1,\ldots,9 \tag{19}$$

Let us consider a large equilibrium deformation (q_i^0, t_k^0) satisfying the equilibrium Eq. (19). Recalling the procedure of small deformations superposed upon large ones, the problem is posed as follows:

Perturbing the controlling parameters t_j^0 so that,

$$t_a = t_a^0 + \mathrm{d}t_a \tag{20}$$

with $|\mathrm{d}t_a| \ll 1$, find the new equilibrium strains v_i of the system with,

$$q_i = q_i^0 + \mathrm{d}q_i \tag{21}$$

in the neighborhood of equilibrium strains q_i^0.

Considering,

$$\mathbf{L} = \left|\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}=\mathbf{q^0}} \tag{22}$$

and following Lazopoulos and Markatis (1996), multiple solutions for the homogeneous deformations may be located when the critical condition,

$$\det \mathbf{L} = 0 \tag{23}$$

Applying principles of branching analysis (Vainberg and Tenogin, 1974; Lazopoulos and Markatis, 1996), the vector $\mathbf{dq} = (\mathrm{d}q_i)$, $i = 1, \ldots, 6$ may be defined by,

$$\mathbf{dq} = \xi \mathbf{dx} + \mathrm{o}(\xi) \tag{24}$$

where, $\mathbf{dx}$ is a solution to the linear equation,

$$\mathbf{L}\,\mathbf{dx} = \mathbf{0} \tag{25}$$

and ξ is defined by the higher order terms of the equilibrium equation, Eq. (19). It is denoted that the singular operator $\mathbf{L}$ denotes loss of strong ellipticity of the stability operator, Eq. (22), see Knowles and Sternberg (1978) for further details.

Moreover, the followed procedure, up to this point, deals with second order transitions, according to Landau et al. (1980) classification. In this case no two-phase deformations are allowed. However, incorporation of the strain jumping conditions introduces additional constraints. Let us point out that the compatibility of the gradient of deformation $\mathbf{F}$ jumping condition is expressed by the existence of a unit vector $\mathbf{f}$ with zero jumping deformation, i.e.

$$[\mathbf{F}] \cdot \mathbf{f} = (\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{f} = \mathbf{0} \tag{26}$$

Recalling Eqs. (15)–(17), (21) and (26) the gradient of deformation for the piece-wise homogeneous deformation is expressed by,

$$\mathbf{F}^{\pm} = \mathbf{F}_0 + \xi^{\pm}\mathbf{F_1} \tag{27}$$

where ξ^+ and ξ^- are ξ parameters of Eq. (24), defined by the higher order terms of the governing equilibrium Eq. (19) and $\mathbf{F_0}$ corresponding to the gradient of deformation in the large equilibrium placement. Furthermore, $\mathbf{F_1}$ depends completely on the kernel $\mathbf{dx}$ of the operator $\mathbf{L}$, defined by Eq. (22). Indeed,

$$\mathbf{F_1} = \begin{vmatrix} \mathrm{d}x_1 & \mathrm{d}x_4 & \mathrm{d}x_5 \\ \mathrm{d}y_1 & \mathrm{d}x_2 & \mathrm{d}x_6 \\ \mathrm{d}y_2 & \mathrm{d}y_3 & \mathrm{d}x_3 \end{vmatrix} \tag{28}$$

where $\mathrm{d}y_i$, $i = 1,2,3$ are linear combinations of $\mathrm{d}x_j$, $j = 1, \ldots ,6$. That is evident recalling Eq. (16) of the conservation of the rotational momentum. Thus, the deformation jumping condition, Eq. (26), requires for two non-zero $\mathbf{f}$ vectors (the phase boundary plane),

$$\frac{\mathrm{d}x_1}{\mathrm{d}y_1}=\frac{\mathrm{d}x_4}{\mathrm{d}x_2}=\frac{\mathrm{d}x_5}{\mathrm{d}x_6}$$
$$\frac{\mathrm{d}x_1}{\mathrm{d}y_2}=\frac{\mathrm{d}x_4}{\mathrm{d}y_3}=\frac{\mathrm{d}x_5}{\mathrm{d}x_3} \tag{29}$$

Eqs. (29) are the jumping compatibility conditions expressed exclusively by the components of the **L** operator. Hence, the existence of a non-zero kernel **dx** of the operator **L**, satisfying further the deformation gradient jumping conditions, is a necessary but not sufficient condition for the two-phase deformation. Recalling the discussion of the emergence of discontinuous strain fields in the one-dimensional case, presented in the preceding section, two phase deformations will be developed in the neighborhood of the cusp singularity, since the cusp is the lowest order cuspoid including Maxwell's sets, required for globally stable transitions.

The existence of a cusp singularity at a point $\mathbf{q^0}$ requires the following conditions:

- (a) The equilibrium condition:

$$\nabla V=\left.\frac{\partial V}{\partial q_i}\right|_{\mathbf{q^0}}=\mathbf{0} \tag{30}$$

- (b) The critical condition:

$$\det\mathbf{L}=0,\quad \text{or}\quad \mathbf{Ldx}=\mathbf{0},\quad \mathbf{dx}\neq\mathbf{0}. \tag{31}$$

- (c) The cusp condition:

$$\left.\frac{\partial^3 V}{\partial q_i\partial q_k\partial q_l}\right|_{\mathbf{q^0}}\mathrm{d}x_i\,\mathrm{d}x_k\,\mathrm{d}x_l=0 \tag{32}$$

which is equivalent to the existence of a six-dimensional vector b_k, $k = 1, \ldots, 6$ satisfying the equation,

$$\left.\frac{\partial^3 V}{\partial q_i\partial q_k\partial q_l}\right|_{\mathbf{q^0}}\mathrm{d}x_k\,\mathrm{d}x_l+\left.\frac{\partial^2 V}{\partial q_i\partial q_k}\right|_{\mathbf{q^0}}b_k=0 \tag{33}$$

In this case, the total potential energy density in the neighborhood of the cusp singularity is expressed by,

$$\begin{aligned}V=&\frac{\xi^4}{4!}\left(\left.\frac{\partial^4 V}{\partial q_i\partial q_k\partial q_l\partial q_r}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}x_k\,\mathrm{d}x_l\,\mathrm{d}x_x+\left.\frac{\partial^3 V}{\partial q_i\partial q_j\partial q_k}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}x_j b_k\right)\\&+\frac{\xi^2}{2}\left(\left.\frac{\partial^3 V}{\partial q_i\partial q_k\partial t_a}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}x_k\,\mathrm{d}t_a+\left.\frac{\partial^3 V}{\partial q_i\partial q_k\partial \mu_b}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}x_k\,\mathrm{d}\mu_b\right)\\&+\xi\left(\left.\frac{\partial^2 V}{\partial q_i\partial t_a}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}t_a+\left.\frac{\partial^2 V}{\partial q_i\partial \mu_b}\right|_{\mathbf{q_i^0}}\mathrm{d}x_i\,\mathrm{d}\mu_b\right)\end{aligned} \tag{34}$$

where, t_a is the vector of the forcing (traction) parameters and μ_b are the constant material parameters.

Thus, the problem has already been reduced to the one-dimensional case and we repeat just the same procedure as in the preceding chapter. The unfolding in the cusp singularity is given by,

$$V=\xi^4-a_0\xi^2+b_0\xi,\quad a_0>0 \tag{35}$$

and the two-phase deformation is defined in Maxwell's set with $b_0 = 0$ and

$$\xi^{\pm} = \sqrt{\frac{2a_0}{3}} \tag{36}$$

Hence, the deformation gradients are expressed by,

$$\mathbf{F}^{\pm} = \mathbf{F_0} + \xi^{\pm}\mathbf{F_1} \tag{37}$$

Let us recall that the phase boundary is defined by the unit vectors **f** of Eq. (26). Further, it is proved that the stress tensor is the same at the two phases. Indeed, the first Piola–Kirchhoff stress tensor is equal to,

$$\mathbf{T}(\mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}} \tag{38}$$

Hence,

$$\mathbf{T}(\mathbf{F}^{\pm}) = \left.\frac{\partial W}{\partial \mathbf{F}}\right|_{\xi=0} + \left.\frac{\partial^2 W}{\partial \mathbf{F}^2}\right|_{\xi=0} \xi^{\pm}\mathbf{F_1} + \frac{1}{2}\left.\frac{\partial^3 W}{\partial \mathbf{F}^3}\right|_{\xi=0} \xi^2\mathbf{F_1^2} + \mathrm{o}(\xi^2) \tag{39}$$

Since,

$$\mathbf{L\,dx} = \left.\frac{\partial^2 W}{\partial \mathbf{F}^2}\right|_{\xi=0} \cdot \mathbf{F_1} = 0 \tag{40}$$

Eq. (53) reveals that

$$T^{+} = T^{-} + \mathrm{o}(\xi^2) \tag{41}$$

Therefore, Eq. (41) covers the equilibrium requirement of the same stress vector at the two phases of the phase boundary. In addition, the total potential energy density function is the same at both phases, because on Maxwell's sets

$$V^{+} = V^{-} \tag{42}$$

Recalling Eq. (14) the Maxwell condition see Gurtin (1983)

$$W^{+} - W^{-} = \mathbf{T}^{\pm}(\mathbf{F}^{+} - \mathbf{F}^{-}) \tag{43}$$

is revealed.

4. Application

Although the procedure is quite general and may be applied to three-dimensional problems and non-isotropic as well, the present application will be restricted to a plane shear problem. The simple problem has been selected just to show the various steps of the method with clarity.

The chosen compressible material is a specific (Blatz and Ko, 1962) material with strain energy density function,

$$W(I_1, J) = I_1 f(J) + g(J) \tag{44}$$

where, I_1 is the first strain invariant and J the determinant of the gradient deformation. Indeed,

$$I_1 = (1 + u_{11})^2 + u_{21}^2 + u_{12}^2 + (1 + u_{22})^2 \tag{45}$$

$$J = \left\{((1 + u_{11})^2 + u_{21}^2)(u_{12}^2 + (1 + u_{22})^2) - (u_{12}^2(1 + u_{11}) + u_{21}(1 + u_{22}))^2\right\}^{1/2} \tag{46}$$

with

$$f(J) = J^{-2} \quad \text{and} \quad g(J) = aJ^2 + bJ^{-1/2} + c \tag{47}$$

Let us notice that no specific difficulty exists in considering any anisotropic strain energy density function. The procedure is just the same.

Since $W(I_1, J)$ has to satisfy zero values and zero stresses at the reference placement, the following relations are valid, see Knowles and Sternberg (1978).

$$\begin{aligned} &2f(1) + g(1) = 0 \\ &2f(1) + f'(1) + g'(1) = 0 \end{aligned} \tag{48}$$

Hence the function g should be,

$$g(J) = aJ^2 + \frac{4(-1+a)}{\sqrt{J}} + 2 - 5a \tag{49}$$

The problem of the initial simple shear in the x_1 direction will be discussed. The emergence of discontinuous deformation gradients will be exhibited and the piece-wise constant strain field will be described. Recalling the strain energy density function

$$W(I_1, J) = I_1 J^{-2} + aJ^2 + \frac{4(-1+a)}{\sqrt{J}} + 2 - 5a \tag{50}$$

the simple shear with the strain components,

$$\begin{aligned} &u_{11} = u_{21} = u_{22} = 0 \\ &u_{12} = k \end{aligned} \tag{51}$$

yields the first Piola–Kirchhoff stress components,

$$\begin{aligned} &t_{11} = t_{22} = 4a - 2(2 + k^2) \\ &t_{12} = 2k \\ &t_{21} = 2k(3 - 2a + k^2) \end{aligned} \tag{52}$$

Furthermore, the four strain components are not independent. The relations expressing the conservation of the rotational momentum, Eqs. (16), yield,

$$u_{11} = -1 + (t_{11}u_{21} + t_{12}(1 + u_{22}) - t_{22}u_{12})/t_{21} = \frac{-k^3 - 2(-1+a)(u_{12} - u_{21}) + k(-2 + 2a + u_{22})}{k(3 - 2a + k^2)} \tag{53}$$

Let us recall the density of the total potential energy for this homogeneous deformation is equal to:

$$V = W - t_{11}u_{11} - t_{12}u_{12} - t_{21}u_{21} - t_{22}u_{22} \tag{54}$$

Since the problem of the initial simple shear is discussed, we try to locate the critical point where the operator **L**, Eq. (22), becomes singular. Recalling Eq. (54), the total energy density V depends on the three strain components u_{11}, u_{21}, u_{22}, i.e.,

$$V = V(u_{12}, u_{21}, u_{22}) \tag{55}$$

Since the pre-critical plane shear deformation is described by Eqs. (51), the post-critical equilibrium path will be described by,

$$\begin{aligned} u_{12} &= k + \xi x_1 \\ u_{21} &= \xi x_2 \\ u_{22} &= \xi x_3 \end{aligned} \tag{56}$$

where, $|\xi| \ll 1$. Yet the total potential energy density $V(u_{12}, u_{21}, u_{22})$ is expanded around the equilibrium state with strains $(u_{12}, u_{21}, u_{22}) = (k, 0, 0)$. Hence,

$$V(u_{12}, u_{21}, u_{22}) = V_0 + \xi V_a + \frac{\xi^2}{2!} V_b + \frac{\xi^3}{3!} V_c + \frac{\xi^4}{4!} V_d + o(\xi^4) \tag{57}$$

The critical strains are defined by the zero second order terms,

$$V_b = 0 \tag{58}$$

Furthermore, the cusp condition requires zero third order terms, i.e.,

$$V_c = 0 \tag{59}$$

Likewise, the involved deformation gradient jumping condition, Eq. (29), is expressed in the present case by,

$$u_{11}u_{22} - (u_{12} - k)u_{21} = 0 \tag{60}$$

Due to conservation of rotational momentum, Eq. (16), u_{11} depends on the other strain coordinates, Eq. (58). Hence recalling the expansion of the strains around the critical point, Eqs(56), the jumping of the strain compatibility condition, Eq. (60), yields,

$$-k^3 x_1 x_2 - 2(-1 + a)(x_1 - x_2)x_3 + k^2(x_1 - x_2)x_3 + k((-3 + 2a)x_1 x_2 + x_3^2) = 0 \tag{61}$$

For (x_1, x_2, x_3) define a direction vector, it may be considered $x_3 = 1$. In this case Eq. (61) may be solved with respect to x_2,

$$x_2 = \frac{-k - 2x_1 + 2ax_1 - k^2 x_1}{2 - 2a + k^2 + 3kx_1 - 2akx_1 + k^3 x_1} \tag{62}$$

with $x_3 = 1$ and x_2 given by Eq. (62). The solution of Eqs. (58), (59) yields the critical $a$and$k$. The component x_1 is also defined by the solution of the system of Eqs. (58) and (59). Using numerical methods with computerized algebra packs, see Mathematica (Wolfram, 1996) the critical x_1, a, k (the x_1^0, a^0, k^0) satisfying Eqs. (58) and (59) have been found equal to,

$$x_1^0 = 1.30, a^0 = 3.442, \text{ and } k^0 = 2.617 \tag{63}$$

Let us remind here that (x_1, x_2, x_3) are the components of the kernel incremental deformation gradient corresponding to the (u_{12}, u_{21}, u_{22}) components, see Eqs. (56).

Recall that the parameter a represents a constitutive parameter, whereas k denotes the shearing. Let us consider incremental values of the controlling parameters, a and k, i.e.,

$$\begin{aligned} a &= a^0 + \delta a \\ k &= k^0 + \delta k \end{aligned} \tag{64}$$

In addition, the computation of the total potential energy, Eq. (34), in the neighborhood of the cusp singularity requires the definition of the vector $b_i, i = 1,3$ introducing second order expansion terms. In the present case,

$$b_1 = 1.43, \quad b_3 = 1 \tag{65}$$

and the total potential energy density has been computed and has found equal to,

$$V = 3.718\xi^4 - 0.222\mathrm{d}k\xi^2 + (1.75\delta a + 0.252\delta k)\xi = 0 \tag{66}$$

Consequently, the problem has been reduced to the one-dimensional case that has already been described in the Section 2. Applying the procedure on global minimization for the one-dimensional case, global minima exist when the ξ coefficient of the total potential, Eq. (66), becomes zero, i.e.

$$1.75\delta a + 0.252\delta k = 0 \tag{67}$$

because the controlling parameters, in that case, are included in Maxwell's set.

Therefore, in the increase δk defined by,

$$\delta k = -6.9\delta a \tag{68}$$

the equilibrium equation yields,

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} = 14.872\xi^3 - 2(0.222\delta k)\xi = 0 \tag{69}$$

with solutions,

$$\xi = 0, \quad \xi = \pm 0.173\delta k^{1/2} \tag{70}$$

Recalling Eq. (62), the critical x_2 may be computed and in fact,

$$x_2^0 = 0.429 \tag{71}$$

Hence, see Eq. (56),

$$\begin{aligned} u_{11} &= 0.56\xi \\ u_{12} &= 2.62 + 1.3\xi \\ u_{21} &= 0.43\xi \\ u_{22} &= \xi \end{aligned} \tag{72}$$

where, ξ is defined by Eq. (69). Besides, the direction of the phase boundary may be defined by the incremental deformation gradient,

$$\mathbf{F_1} = \xi \begin{bmatrix} 0.56 & 1.3 \\ 0.43 & 1 \end{bmatrix} \tag{73}$$

see Eq. (27). The unit vector $\mathbf{f}$ describing the phase boundary is given by the equation:

$$\mathbf{F}_1\mathbf{f} = \mathbf{0} \tag{74}$$

In the present case, the unit vector $\mathbf{f}$ directed parallel to the phase boundary is found to be equal to

$$\mathbf{f} = (-0.91, 0.39)^{\mathrm{T}} \tag{75}$$

Concluding, two-phase deformations for the present problem may be emerged when the constitutive parameter a is equal to,

$$a = 3.442 + \delta a, \quad |\delta a| \ll 1, \ \ \delta\alpha < 0 \tag{76}$$

and the shearing k,

$$k = 2.617 - 6.9\delta a \quad |\delta a| \ll 1, \ \ \delta\alpha < 0 \tag{77}$$

Consequently, if the Piola–Kirchhoff traction equals,

$$\begin{aligned} t_{11} &= t_{22} = -3.929 + 76.22\delta a \\ t_{12} &= 5.234 - 13.8\delta a \\ t_{21} &= 15.52 - 240.4\delta a \end{aligned} \tag{78}$$

and is applied to the specific material, two phase deformations are shown up.

The two phases are defined by the displacement gradient components,

$$\begin{aligned} u_{11} &= \pm 0.2272(-\delta a)^{1/2} \\ u_{12} &= 2.62 \pm 0.5909(-\delta a)^{1/2} \\ u_{21} &= \pm 0.1953(-\delta a)^{1/2} \\ u_{22} &= \pm 0.4544(-\delta a)^{1/2} \end{aligned} \tag{79}$$

The plus sign corresponds to one phase, whereas the minus sign to the other phase. The phase boundary aligns along the direction of the unit vector $(-0.91, 0.39)$.

Therefore, the discontinuous deformation gradient strain field has completely been defined.

5. Conclusion

A general procedure for the description of two-phase fields in homogeneous deformations in finite elasticity has been proposed. The procedure is based upon singularity theory. It has been found that bifurcation is a necessary condition for emergence of discontinuous strains in (piece-wise) homogeneous deformations. Nevertheless it is not sufficient. The deformation gradient jumping compatibility condition restricts the kernel space of the branching problem. Furthermore, globally stable transitions, requiring multiple global minima, are shown up if the cusp condition for the total potential energy density function is satisfied. In fact the existence of Maxwell's set, allowing for multiple global minima, require at least the cusp condition for the total potential energy function. Consequently the branching critical condition should be combined with the strain jumping and cusp conditions for the emergence of discontinuous strain fields. The present procedure may be applied to any anisotropic material under any homogeneous deformation. The method works in three-dimensional problems. The method may be extended including materials with internal constraints.

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